SUGGESTED SOLUTIONS TO HOMEWORK 4

Exercise 1 (6.4.17)**.** Suppose that $I \subseteq \mathbb{R}$ is an open interval and that $f''(x) \geq 0$ for all $x \in I$. If $c \in I$, show that the part of the graph of f on I is never below the tangent line to the graph at $(c, f(c))$.

Proof. By Taylor's Theorem, for any $x \in I$, there exists a $x_0 \in I$ between c and x such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(x_0)}{2}(x-c)^2,$$

which implies that

$$f(x) \geq f(c) + f'(c)(x-c),$$

since $f''(x) \geq 0$ for all $x \in I$. Therefore the part of the graph of f on I is never below the tangent line to the graph at $(c, f(c))$. □

Exercise 2 (6.4.18)**.** Let $I \subseteq \mathbb{R}$ be an interval and $c \in I$. Suppose that f and g are defined on I and that the derivatives $f^{(n)}$, $g^{(n)}$ exist and are continuous on I. If $f^{(k)}(c) = 0$ and $g^{(k)}(c) = 0$ for $k = 0, 1, ..., n-1$, but $g^{(n)}(c) \neq 0$, show that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

Proof. Without loss of generality, we assume $g^{(n)}(c) > 0$. By continuity of $g^{(n)}$, there exists a $\delta_1 > 0$ such that

$$|g^{(n)}(x) - g^{(n)}(c)| < \frac{1}{2}g^{(n)}(c),$$

for all $|x - c| < \delta_1$, which implies

$$g^{(n)}(x) > \frac{1}{2}g^{(n)}(c),$$

for all $|x - c| < \delta_1$. Moreover, By continuity of $f^{(n)}$ and $g^{(n)}$, for arbitrary $\varepsilon > 0$, there exists a $\delta_2 > 0$ such that

$$|f^{(n)}(x) - f^{(n)}(c)| < \frac{|g^{(n)}(c)|}{4}\varepsilon, \quad |g^{(n)}(x) - g^{(n)}(c)| < \frac{|g^{(n)}(c)|^2}{4|f^{(n)}(c)|}\varepsilon,$$

for all $|x - c| < \delta_2$. Then by Taylor's Theorem, for all $|x - c| < \min\{\delta_1, \delta_2\}$, we have

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{f^{(n)}(c)}{g^{(n)}(c)}\right| &= \left|\frac{f^{(n)}(x_0)}{g^{(n)}(x_0)} - \frac{f^{(n)}(c)}{g^{(n)}(c)}\right| \\
&\leq \frac{2}{|g^{(n)}(c)|^2}\left(|f^{(n)}(x_0) - f^{(n)}(c)||g^{(n)}(c)| + |f^{(n)}(c)||g^{(n)}(x_0) - g^{(n)}(c)|\right) \\
&< \varepsilon,
\end{aligned}$$

which implies that

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f^{(n)}(c)}{g^{(n)}(c)}.$$

□

Exercise 3 (6.4.19)**.** Show that the function $f(x) := x^3 - 2x - 5$ has a zero r in the interval $I := [2, 2.2]$. If $x_1 := 2$ and if we define the sequence (x_n) using the Newton procedure, show that $|x_{n+1} - r| \leq (0.7)|x_n - r|^2$. Show that x_4 is accurate to within six decimal places.

Proof. Since

$$f(2)f(2.2) = -1.248 < 0,$$

which implies that f has a zero in I. In addition, since

$$|f'(x)| = |3x^2 - 2| \geq 10, \quad x \in I,$$
$$|f''(x)| = |6x| \leq 13.2, \quad x \in I.$$

By Newton's Method, we have

$$K = \frac{13.2}{20} = 0.66,$$

which implies that

$$|x_{n+1} - r| \leq 0.7|x_n - r|^2.$$

Therefore

$$|x_4 - r| \leq 0.7|x_3 - r|^2 \leq 0.7^2|x_2 - r|^4 \leq 0.7^3|x_1 - r|^8 \leq 9 \times 10^{-7},$$

which implies x_4 is accurate to within six decimal places. □

Exercise 4 (6.4.21)**.** Approximate the real zeros of $h(x) := x^3 - x - 1$. Apply Newton's Method starting with the initial choices (a) $x_1 := 2$, (b) $x_1 := 0$, (c) $x_1 := -2$. Explain what happens.

Proof. Since

$$h'(x) = 3x^2 - 1,$$

the iteration formula is

$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1} = \frac{2x_n^3 + 1}{3x_n^2 - 1}.$$

(a) For $x_1 := 2$, we have $x_2 = 1.5454545454$, $x_3 = 1.359614915915184$, $x_4 = 1.325801345005845$, $x_5 = 1.3247190494171253$, $x_6 = 1.3247179572458576$.

(b) For $x_1 := 0$, we have $x_2 = -1$, $x_3 = -0.5$, $x_4 = -3$, $x_5 = -2.0384615384615383$, $x_6 = -1.3902821472167362$, $x_7 = -0.9116118977179269$, $x_8 = -0.34502849674816904$, $x_9 = -1.42775070402727$, $x_{10} = -0.9424179125094826$, $x_{11} = -0.4049493571993791$, $x_{12} = -1.7069046451828482$, $x_{13} = -1.1557563610748112$, $x_{14} = -0.6941918133295446$, $x_{15} = 0.7424942987207317$, $x_{16} = 2.7812959406771816$, $x_{17} = 1.9827252470435746$, $x_{18} = 1.5369273797581307$, $x_{19} = 1.3572624831876983$, $x_{20} = 1.3256630944288657$, $x_{21} = 1.324718788615257$, $x_{22} = 1.32471795724539$.

(c) For $x_1 := -2$, we have $x_2 = -1.3636363636363638$, $x_3 = -0.8892353134230391$, $x_4 = -0.29609491722682657$, $x_5 = -1.2864352364252927$, $x_6 = -0.8217128731113184$, $x_7 = -0.10691984835296453$, $x_8 = -1.03298211799929$, $x_9 = -0.5472085094190648$, $x_{10} = -6.611274711607757$, $x_{11} = -4.433702585989313$, $x_{12} = -2.989538057557567$, $x_{13} = -2.031496813533126$, $x_{14} = -1.3854649531667529$, $x_{15} = -0.9075970552862211$, $x_{16} = -0.33661987193684095$, $x_{17} = -1.3994357597139493$, $x_{18} = -0.9192055355939536$, $x_{19} = -0.3605283713478141$, $x_{20} = -1.4855582794693762$, $x_{21} = -0.9886592503827742$, $x_{22} = -0.4826912520186133$, $x_{23} = -2.5747641103691308$, $x_{24} = -1.7544435797503293$, $x_{25} = -1.190229561606389$, $x_{26} = -0.7299425321629264$, $x_{27} = 0.37120956464817956$,

$x_{28} = -1.8791055025333052$, $x_{29} = -1.2790826744313537$, $x_{30} = -0.8150369612962106$, $x_{31} = -0.08343010635041948$, $x_{32} = -1.0201408804244034$, $x_{33} = -0.5293414717824825$, $x_{34} = -4.4127113786720065$, $x_{35} = -2.9756275206939176$, $x_{36} = -2.0222349844904874$, $x_{37} = -1.3790536473294959$, $x_{38} = -0.9022331545785207$, $x_{39} = -0.3251428738220351$, $x_{40} = -1.3637814489635551$, $x_{41} = -0.8893583038994928$, $x_{42} = -0.2963784747231665$, $x_{43} = -1.2871128668662455$, $x_{44} = -0.822325743167123$, $x_{45} = -0.10902119046333758$, $x_{46} = -1.0342878895057943$, $x_{47} = -0.5489915337873275$, $x_{48} = -6.9822885227693545$, $x_{49} = 4.680020332909626$, $x_{50} = -3.1527764448637754$, $x_{51} = -2.140083129871087$, $x_{52} = -1.4602171044569312$, $x_{53} = -0.9685635995638602$, $x_{54} = -0.4504371240959939$, $x_{55} = -2.0883677454698484$, $x_{56} = -1.4247054725546997$, $x_{57} = -0.9399406438426444$, $x_{58} = -0.40040427420864433$, $x_{59} = -1.6793109477226182$, $x_{60} = -1.1355642957994752$, $x_{61} = -0.6723450974394425$, $x_{62} = 1.1010597171365635$, $x_{63} = 1.3916209468034497$,
$x_{64} = 1.3285408122104765$, $x_{65} = 1.3247315055694269$, $x_{66} = 1.3247179574157972$.

Recall that

$$h'(x) = 3x^2 - 1 = 3\left(x + \frac{\sqrt{3}}{3}\right)\left(x - \frac{\sqrt{3}}{3}\right),$$

which implies that h is increasing over $(-\infty, -\sqrt{3}/3)$, decreasing over $(-\sqrt{3}/3, \sqrt{3}/3)$ and increasing over $(\sqrt{3}/3, \infty)$. The local maximum is $h\left(-\sqrt{3}/3\right) = 2\sqrt{3}/9 - 1 < 0$ and $\lim_{x\to\infty} h(x) = \infty$, therefore h has a unique root (three roots with the same value) r. Since $h'(\sqrt{3}/3) = 0$ and $h\left(\sqrt{3}/3\right) = -2\sqrt{3}/9 - 1 < 0$, by the convergence theorem for Newton's Method, the initial approximate value x_1 should at least lie in $(\sqrt{3}/3, \infty)$, which implies that Newton's Method is not suitable to be apply in case (b) and (c). This also explains the huge amount of computation in case (b) and (c).

Set $I := [1.32, 2]$. Since $f(1.32)f(2) < 0$, we have $r \in I$. Moreover,

$$|f'(x)| \geq |f'(1.32)| = 4.2271, \quad x \in I,$$
$$|f''(x)| \leq |f''(2)| = 12, \quad x \in I,$$

then by the convergence theorem for Newton's Method,

$$K = \frac{6}{4.2271} = 1.419379258137774,$$

therefore the initial approximate value x_0 satisfies

$$|x_0 - r| < \frac{1}{K} = 0.7045333333333332.$$

Because $1.32 + 0.7045333333333332 > 2$ and $2 \in I$, the convergence theorem for Newton's theorem can be applied to case (a). This explains the fast convergence in case (a). □

Exercise 5 (7.1.1)**.** If $I := [0, 4]$, calculate the norms of the following partitions:
(a) $\mathcal{P} := (0, 1, 2, 4)$,
(b) $\mathcal{P} := (0, 2, 3, 4)$,
(c) $\mathcal{P} := (0, 1, 1.5, 2, 3.4, 4)$,
(d) $\mathcal{P} := (0, .5, 2.5, 3.5, 4)$.

Proof. (a) $\|\mathcal{P}\| = 2$.
(b) $\|\mathcal{P}\| = 2$.
(c) $\|\mathcal{P}\| = 1.4$.
(d) $\|\mathcal{P}\| = 2$. □

Exercise 6 (7.1.2)**.** If $f(x) := x^2$ for $x \in [0,4]$, calculate the following Riemann sums, where $\dot{\mathcal{P}}_i$ has the same partition points as in Exercise 5, and the tags are selected as indicated.

(a) $\dot{\mathcal{P}}_1$ with the tags at the left endpoints of the subintervals.
(b) $\dot{\mathcal{P}}_1$ with the tags at the right endpoints of the subintervals.
(c) $\dot{\mathcal{P}}_2$ with the tags at the left endpoints of the subintervals.
(d) $\dot{\mathcal{P}}_2$ with the tags at the right endpoints of the subintervals.

Proof. (a) $S(f, \dot{\mathcal{P}}) = 9$.
(b) $S(f, \dot{\mathcal{P}}) = 37$.
(c) $S(f, \dot{\mathcal{P}}) = 13$.
(d) $S(f, \dot{\mathcal{P}}) = 33$. $\square$